13-2 Angles and Angle Measure

Main Ideas

- Change radian measure to degree measure and vice versa.
- Identify coterminal angles.

New Vocabulary

initial side
terminal side
standard position
unit circle
radian
coterminal angles

Reading Math

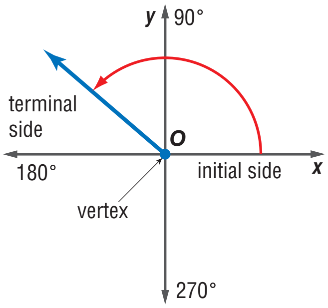
Angle of Rotation In trigonometry, an angle is sometimes referred to as an *angle of rotation.*

GET READY for the Lesson

The Ferris wheel at Navy Pier in Chicago has a 140-foot diameter and 40 gondolas equally spaced around its circumference. The average angular velocity ω of one of the gondolas is given by $\omega = \frac{\theta}{t}$ where θ is the angle through which the gondola has revolved after a specified amount of time t. For example, if a gondola revolves through an angle of 225° in 40 seconds, then its average angular velocity is 225° ÷ 40 or about 5.6° per second.

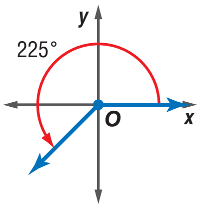
ANGLE MEASUREMENT What does an angle measuring 225° look like? In Lesson 13-1, you worked only with acute angles, those measuring between 0° and 90°, but angles can have *any* real number measurement.

On a coordinate plane, an angle may be generated by the rotation of two rays that share a fixed endpoint at the origin. One ray, called the **initial side** of the angle, is fixed along the positive x-axis. The other ray, called the **terminal side** of the angle, can rotate about the center. An angle positioned so that its vertex is at the origin and its initial side is along the positive x-axis is said to be in **standard position**.

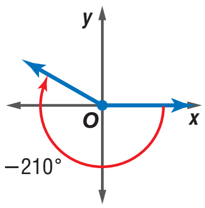


The measure of an angle is determined by the amount and direction of rotation from the initial side to the terminal side.

Positive Angle Measure
counterclockwise



Negative Angle Measure
clockwise



Concepts in Motion
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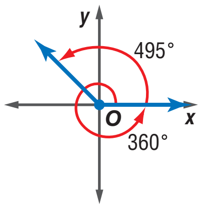
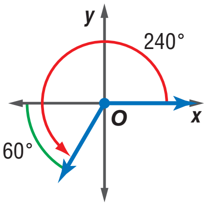
When terminal sides rotate, they may sometimes make one or more revolutions. An angle whose terminal side has made exactly one revolution has a measure of 360°.



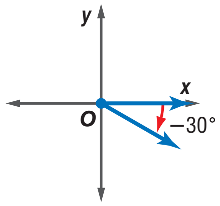
EXAMPLE Draw an Angle in Standard Position

1 **Draw an angle with the given measure in standard position.**

a. 240° $240° = 180° + 60°$
Draw the terminal side of the angle 60° counterclockwise past the negative x-axis.



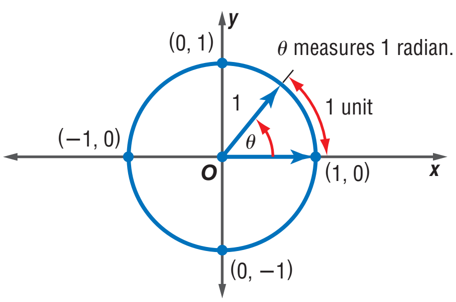
b. −30° The angle is negative.
Draw the terminal side of the angle 30° clockwise from the positive x-axis.

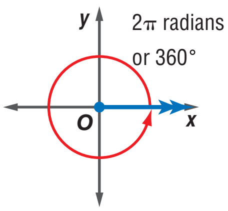


CHECK Your Progress

1A. 450° **1B.** −110°

Another unit used to measure angles is a radian. The definition of a radian is based on the concept of a **unit circle**, which is a circle of radius 1 unit whose center is at the origin of a coordinate system. One **radian** is the measure of an angle θ in standard position whose rays intercept an arc of length 1 unit on the unit circle.





The circumference of any circle is $2\pi r$, where r is the radius measure. So the circumference of a unit circle is $2\pi(1)$ or 2π units. Therefore, an angle representing one complete revolution of the circle measures 2π radians. This same angle measures 360°. Therefore, the following equation is true.

$$2\pi \text{ radians} = 360°$$

As with degrees, the measure of an angle in radians is positive if its rotation is counterclockwise. The measure is negative if the rotation is clockwise.

To change angle measures from radians to degrees or vice versa, solve the equation above in terms of both units.

$$2\pi \text{ radians} = 360°$$
$$\frac{2\pi \text{ radians}}{2\pi} = \frac{360°}{2\pi}$$
$$1 \text{ radian} = \frac{180°}{\pi}$$

1 radian is about 57 degrees.

$$2\pi \text{ radians} = 360°$$
$$\frac{2\pi \text{ radians}}{360} = \frac{360°}{360}$$
$$\frac{\pi \text{ radians}}{180} = 1°$$

1 degree is about 0.0175 radian.

These equations suggest a method for converting between radian and degree measure.

Reading Math

Radian Measure The word *radian* is usually omitted when angles are expressed in radian measure. Thus, when no units are given for an angle measure, radian measure is implied.

KEY CONCEPT — *Radian and Degree Measure*

- To rewrite the radian measure of an angle in degrees, multiply the number of radians by $\frac{180°}{\pi \text{ radians}}$.
- To rewrite the degree measure of an angle in radians, multiply the number of degrees by $\frac{\pi \text{ radians}}{180°}$.

EXAMPLE — Convert Between Degree and Radian Measure

2 **Rewrite the degree measure in radians and the radian measure in degrees.**

a. **60°**

$$60° = 60\cancel{°}\left(\frac{\pi \text{ radians}}{180\cancel{°}}\right)$$
$$= \frac{60\pi}{180} \text{ or } \frac{\pi}{3} \text{ radians}$$

b. $-\frac{7\pi}{4}$

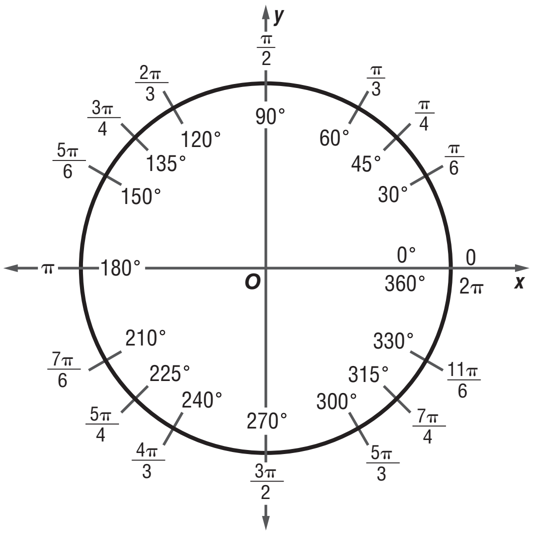
$$-\frac{7\pi}{4} = \left(-\frac{7\cancel{\pi}}{4}\ \cancel{\text{radians}}\right)\left(\frac{180°}{\cancel{\pi \text{ radians}}}\right)$$
$$= -\frac{1260°}{4} \text{ or } -315°$$

CHECK Your Progress

2A. **190°**

2B. $\frac{3\pi}{8}$

You will find it useful to learn equivalent degree and radian measures for the special angles shown in the diagram at the right. This diagram is more easily learned by memorizing the equivalent degree and radian measures for the first quadrant and for 90°. All of the other special angles are multiples of these angles.



Concepts in Motion

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EXAMPLE Measure an Angle in Degrees and Radians

3 **TIME** **Find both the degree and radian measures of the angle through which the hour hand on a clock rotates from 1:00 P.M. to 3:00 P.M.**

The numbers on a clock divide it into 12 equal parts with 12 equal angles. The angle from 1 to 3 on the clock represents $\frac{2}{12}$ or $\frac{1}{6}$ of a complete rotation of 360°. $\frac{1}{6}$ of 360° is 60°.

Since the rotation is clockwise, the angle through which the hour hand rotates is negative. Therefore, the angle measures $-60°$.

60° has an equivalent radian measure of $\frac{\pi}{3}$. So the equivalent radian measure of $-60°$ is $-\frac{\pi}{3}$.

Real-World Link

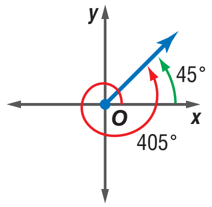
The clock tower in the United Kingdom Parliament House was opened in 1859. The copper minute hand in each of the four clocks of the tower is 4.2 meters long, 100 kilograms in mass, and travels a distance of about 190 kilometers a year.

Source: parliament.uk/index.cfm

CHECK Your Progress

3. How long does it take for a minute hand on a clock to pass through 2.5π radians?

COTERMINAL ANGLES If you graph a 405° angle and a 45° angle in standard position on the same coordinate plane, you will notice that the terminal side of the 405° angle is the same as the terminal side of the 45° angle. When two angles in standard position have the same terminal sides, they are called **coterminal angles**.



Notice that $405° - 45° = 360°$. In degree measure, coterminal angles differ by an integral multiple of 360°. You can find an angle that is coterminal to a given angle by adding or subtracting a multiple of 360°. In radian measure, a coterminal angle is found by adding or subtracting a multiple of 2π.

EXAMPLE Find Coterminal Angles

4 **Find one angle with positive measure and one angle with negative measure coterminal with each angle.**

a. **240°**

A positive angle is $240° + 360°$ or $600°$.

A negative angle is $240° - 360°$ or $-120°$.

b. $\frac{9\pi}{4}$

A positive angle is $\frac{9\pi}{4} + 2\pi$ or $\frac{17\pi}{4}$. $\quad \frac{9\pi}{4} + \frac{8\pi}{4} = \frac{17\pi}{4}$

A negative angle is $\frac{9\pi}{4} - 2(2\pi)$ or $-\frac{7\pi}{4}$. $\quad \frac{9\pi}{4} + \left(\frac{-16\pi}{4}\right) = \frac{-7\pi}{4}$

Study Tip

Coterminal Angles

Notice in Example 4b that it is necessary to subtract a multiple of 2π to find a coterminal angle with negative measure.

CHECK Your Progress

4A. 15° **4B.** $-\frac{\pi}{4}$

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CHECK Your Understanding

Example 1 (p. 769)

Draw an angle with the given measure in standard position.

1. $70°$ **2.** $300°$ **3.** $570°$ **4.** $-45°$

Example 2 (p. 770)

Rewrite each degree measure in radians and each radian measure in degrees.

5. $130°$ **6.** $-10°$ **7.** $485°$

8. $\frac{3\pi}{4}$ **9.** $-\frac{\pi}{6}$ **10.** $\frac{19\pi}{3}$

Example 3 (pp. 770–771)

ASTRONOMY **For Exercises 11 and 12, use the following information.**
Earth rotates on its axis once every 24 hours.

11. How long does it take Earth to rotate through an angle of 315°?

12. How long does it take Earth to rotate through an angle of $\frac{\pi}{6}$?

Example 4 (p. 771)

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

13. $60°$ **14.** $425°$ **15.** $\frac{\pi}{3}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
16–19	1
20–27	2
28–33	4
34, 35	3

Draw an angle with the given measure in standard position.

16. $235°$ **17.** $270°$ **18.** $790°$ **19.** $380°$

Rewrite each degree measure in radians and each radian measure in degrees.

20. $120°$ **21.** $60°$ **22.** $-15°$ **23.** $-225°$

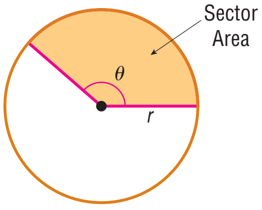
24. $\frac{5\pi}{6}$ **25.** $\frac{11\pi}{4}$ **26.** $-\frac{\pi}{4}$ **27.** $-\frac{\pi}{3}$

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

28. $225°$ **29.** $30°$ **30.** $-15°$

31. $\frac{3\pi}{4}$ **32.** $\frac{7\pi}{6}$ **33.** $-\frac{5\pi}{4}$

GEOMETRY **For Exercises 34 and 35, use the following information.**
A *sector* is a region of a circle that is bounded by a central angle θ and its intercepted arc. The area A of a sector with radius r and central angle θ is given by $A = \frac{1}{2}r^2\theta$, where θ is measured in radians.



34. Find the area of a sector with a central angle of $\frac{4\pi}{3}$ radians in a circle whose radius measures 10 inches.

35. Find the area of a sector with a central angle of 150° in a circle whose radius measures 12 meters.

Draw an angle with the given measure in standard position.

36. $-150°$ **37.** $-50°$ **38.** π **39.** $-\frac{2\pi}{3}$

Rewrite each degree measure in radians and each radian measure in degrees.

40. 660° **41.** 570° **42.** 158° **43.** 260°

44. $\frac{29\pi}{4}$ **45.** $\frac{17\pi}{6}$ **46.** 9 **47.** 3

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

48. −140° **49.** 368° **50.** 760°

51. $-\frac{2\pi}{3}$ **52.** $\frac{9\pi}{2}$ **53.** $\frac{17\pi}{4}$

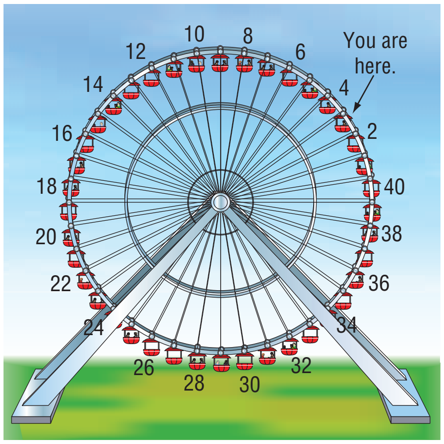
54. DRIVING Some sport-utility vehicles (SUVs) use 15-inch radius wheels. When driven 40 miles per hour, determine the measure of the angle through which a point on the wheel travels every second. Round to both the nearest degree and the nearest radian.

Real-World Link

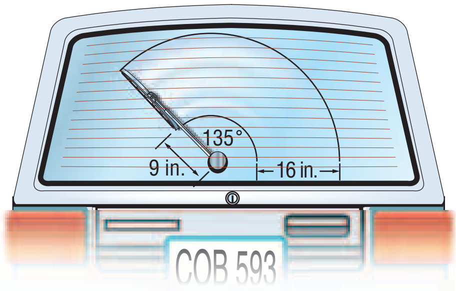
Vehicle tires are marked with numbers and symbols that indicate the specifications of the tire, including its size and the speed the tire can safely travel.

Source: usedtire.com

55. ENTERTAINMENT Suppose the gondolas on the Navy Pier Ferris Wheel were numbered from 1 through 40 consecutively in a counterclockwise fashion. If you were sitting in gondola number 3 and the wheel were to rotate counterclockwise through $\frac{47\pi}{10}$ radians, which gondola used to be in the position that you are in now?



56. CARS Use the Area of a Sector Formula in Exercises 34 and 35 to find the area swept by the rear windshield wiper of the car shown at the right.



EXTRA PRACTICE
See pages 920, 938.

Math Online
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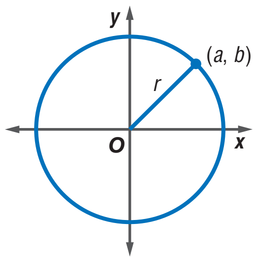
H.O.T. Problems

57. OPEN ENDED Draw and label an example of an angle with negative measure in standard position. Then find an angle with positive measure that is coterminal with this angle.

58. CHALLENGE A line with positive slope makes an angle of $\frac{\pi}{2}$ radians with the positive x-axis at the point (2, 0). Find an exact equation for this line.

59. CHALLENGE If (a, b) is on a circle that has radius r and center at the origin, prove that each of the following points is also on this circle.

a. $(a, -b)$ **b.** (b, a) **c.** $(b, -a)$



60. REASONING Express $\frac{1}{8}$ of a revolution in degrees.

61. *Writing in Math* Use the information on page 768 to explain how angles can be used to describe circular motion. Include an explanation of the significance of angles of more than 180° in terms of circular motion, an explanation of the significance of angles with negative measure in terms of circular motion, and an interpretation of a rate of more than 360° per minute.

STANDARDIZED TEST PRACTICE

62. **ACT/SAT** Choose the radian measure that is equal to 56°.

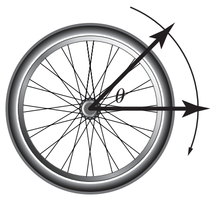
A $\frac{\pi}{15}$

B $\frac{7\pi}{45}$

C $\frac{14\pi}{45}$

D $\frac{\pi}{3}$

63. **REVIEW** Angular velocity is defined by the equation $\omega = \frac{\theta}{t}$, where θ is usually expressed in radians and t represents time. Find the angular velocity in radians per second of a point on a bicycle tire if it completes 2 revolutions in 3 seconds.



F $\frac{\pi}{3}$ H $\frac{2\pi}{3}$

G $\frac{\pi}{2}$ J $\frac{4\pi}{3}$

Spiral Review

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-1)

64. $A = 34°, b = 5$

65. $B = 68°, b = 14.7$

66. $B = 55°, c = 16$

67. $a = 0.4, b = 0.4\sqrt{3}$

Find the margin of sampling error. (Lesson 12-9)

68. $p = 72\%, n = 100$

69. $p = 50\%, n = 200$

Determine whether each situation involves a *permutation* or a *combination*. Then find the number of possibilities. (Lesson 12-2)

70. choosing an arrangement of 5 CDs from your 30 favorite CDs

71. choosing 3 different types of snack foods out of 7 at the store to take on a trip

Find $[g \circ h](x)$ and $[h \circ g](x)$. (Lesson 7-1)

72. $g(x) = 2x$
$h(x) = 3x - 4$

73. $g(x) = 2x + 5$
$h(x) = 2x^2 - 3x + 9$

GET READY for the Next Lesson

PREREQUISITE SKILL **Simplify each expression.** (Lesson 7-5)

74. $\frac{2}{\sqrt{3}}$

75. $\frac{3}{\sqrt{5}}$

76. $\frac{4}{\sqrt{6}}$

77. $\frac{5}{\sqrt{10}}$

78. $\frac{\sqrt{7}}{\sqrt{2}}$

79. $\frac{\sqrt{5}}{\sqrt{8}}$